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An Operator Approach to Tangent Vector Fields Processing Supplemental Material

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2. Vector Fields as Operators

Lemma 2.1 Let V a vector field on M and let T_F^t, $t \in \mathbb{R}$ be the functional representations of the diffeomorphisms $\Phi_V^t : M \to M$ of the one parameter group associated to the flow of V. If D is a linear partial differential operator then $D_V \circ D = D \circ D_V$ if and only if for any $t \in \mathbb{R}$, $T_F^t \circ D = D \circ T_F^t$.

Proof Let $p \in M$ and $f \in C^\infty(M)$ be a smooth function. If $V(p) = 0$, then $\Phi_V^t(p) = p$ and $D_V(f)(p) = 0$. It immediately follows that $D_V \circ D(f)(p) = D \circ D_V(f)(p)$ if and only if $T_F^t \circ D(f)(p) = D \circ T_F^t(f)(p)$ because the right hand side of both equation is equal to 0.

Now assume that $V(p) \neq 0$. There exists (see, e.g. [Spi99] Theorem 7, p.148) a local coordinate system in an open neighborhood of p such that $V = \frac{\partial}{\partial x}$ and D can be written as

$$D = \sum_{0<|\alpha|\leq n} a_\alpha(x,y)\partial^\alpha$$

where $\alpha = (i, j)$ is a multi-index , $|\alpha| = i + j$ and $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x^i \partial x^j}$.

First assume that $T_F^t \circ D = D \circ T_F^t$. Since the derivative (with respect to t) of $f \circ \Phi_V^t(p)$ at $t = 0$ is equal to $D_V(f)(p)$, the differentiation with respect to t of the equality $D(f)(\Phi_V^t(p)) = D(f \circ \Phi_V^t(p))$ gives at $t = 0$: $D_V(D(f))(p) = D(D_V(f))(p)$. As this holds for any f and p, we deduce that $D_V \circ D = D \circ D_V$.

Assume now that $D_V \circ D = D \circ D_V$. As in the proof of Lemma 2.4, since the flow of V is a one parameter group we just need to prove that $T_F^t \circ D = D \circ T_F^t$ for t contained in an arbitrarily small interval containing 0 but not reduced to 0. Using the product rule we have

$$0 = D_V \circ D(f) - D(D_V(f)) \;=\; \sum_{0<|\alpha=(i,j)|\leq n} \frac{\partial a_\alpha}{\partial x} \frac{\partial^\alpha f}{\partial x^i \partial x^j}.$$

Since this equality holds for any f we deduce that for any α, $\frac{\partial a_\alpha}{\partial x} = 0$. As a consequence, the coefficients a_α of D are constant along the trajectories of V in the local coordinate system and thus for $|t|$ small enough we obtain $T_F^t \circ D(f)(p) = D \circ T_F^t(f)(p)$. □

Lemma 2.2 A vector field V is a Killing vector field if and only if $D_V \circ L = L \circ D_V$.

Proof As L is a differential operator, it follows from Lemma 2.1 that $D_V \circ L = L \circ D_V$ if and only if $T_F^t \circ L = L \circ T_F^t$. Recalling that the Laplace-Beltrami operator is invariant under the action of isometries of M, we immediately deduce that if V is a Killing vector field then $D_V \circ L = L \circ D_V$. Now, if $T_F^t \circ L = L \circ T_F^t$, then the Laplace-Beltrami operator L is preserved by the action of the diffeomorphims Φ_V^t. Since L determines the metric on M, Φ_V^t have to be isometries. □

Lemma 2.3 Given two vector fields D_{V_1} and D_{V_2} that both commute with some operator D, the Lie derivative $\mathcal{L}_{V_1}(V_2)$ will also commute with D.

Proof Using that $DD_{V_1} = D_{V_1}D$ and $DD_{V_2} = D_{V_2}D$ we immediately obtain

$$\begin{aligned} D(D_{V_1}D_{V_2} - D_{V_2}D_{V_1}) &= DD_{V_1}D_{V_2} - DD_{V_2}D_{V_1} \\ &= D_{V_1}D_{V_2}D - D_{V_2}D_{V_1}D \\ &= (D_{V_1}D_{V_2} - D_{V_2}D_{V_1})D. \end{aligned}$$

□

Lemma 2.4 $D_{V_2} = (T_F)^{-1} \circ D_{V_1} \circ T_F$.

Proof Given $p \in M$, by definition of the push forward we have $V_2(T(p)) = dT(V_1(p))$ where dT denotes the differential of the diffeomorphism T. Now if $f \in C^\infty(N)$ is a smooth function, then using the chain rule we get

$$\begin{aligned} D_{V_1} \circ T_F(f)(p) = D_{V_1}(f \circ T)(p) &= d(f \circ T)(V_1(p)) \\ &= df(dT(V_1(p))) \\ &= df(V_2(T(p))) \\ &= D_{V_2}(f)(T(p)) \\ &= T_F \circ D_{V_2}(f)(p) \end{aligned}$$

As T is a diffeomorphism, T_F is an isomorphism and we obtain $D_{V_2} = (T_F)^{-1} \circ D_{V_1} \circ T_F$. □

Lemma 2.5 Assume that the manifold M and the vector field V are real analytic. Let $T^t = \Phi_V^t$ be self-map associated with the flow of V at time t. Then if T_F^t is the functional representation of T^t, for any real analytic function f:

$$T_F^t f = \exp(t\, D_V) f = \sum_{k=0}^{\infty} \frac{(tD_V)^k f}{k!}.$$

Proof The set of diffeomorphisms associated to the flow of V is a one parameter group: for $t,s \in \mathbb{R}$, $\Phi_V^{t+s} = \Phi_V^t \circ \Phi_V^s$ (see [Spi99], Theorem 6, p.147). The right hand side of the equality of the Lemma also having the same property, it sufficies to show it for t contained in any arbitrarily small interval containing 0 but not reduced to 0. Given $p \in M$, if $V(p) = 0$, then for any k, $(D_V)^k(f)(p) = 0$ and both hand sides of the equality are equal to $f(p)$. Now assume that $V(p) \neq 0$. There exists (see, e.g. [Spi99] Theorem 7, p.148) an analytic local coordinate system in an open neighborhod of p in which V is equal to $\frac{\partial}{\partial x}$. As a consequence without loss of generality we can assume that $V = \frac{\partial}{\partial x}$ and $p = 0$, and prove the equality in this coordinate system. As the flow of $\frac{\partial}{\partial x}$ is just a translation, the left hand side of the equality becomes $T_F f(0) = f(t)$. As $D_{\frac{\partial}{\partial x}}(f) = \frac{\partial f}{\partial x}$, the right hand side is just the Taylor expansion of f at 0 in the direction of x:

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^k f}{\partial x^k}(0).$$

Since f is an analytic function, for $|t|$ small enough, this Taylor expansion is equal to $f(t)$. □

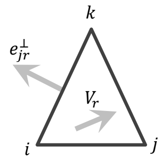
4. Discretization

4.1. Derivation of the discrete operator

To compute the entries in the matrix S, we need to compute integrals of the form $d_{ij}^r = \int_{t_r} \gamma_i \langle \nabla\gamma_j, V_r \rangle d\mu$, where t_r is a triangle, γ_i is the hat basis function of the vertex i, and V_r is a constant vector in t_r. These integrals are non zero only if both i and j are vertices of t_r, and their value is given by the following Lemma.

Lemma 4.0 Let $M = (X,F,N)$ and let V be a piecewise constant vector field on M. In addition, let $t_r = (i,j,k) \in F$ be a triangle and V_r be the value of V on t_r. Then:

$$d_{ij}^r = \int_{t_r} \gamma_i \langle \nabla\gamma_j, V_r \rangle d\mu = \frac{1}{6} \left\langle e_{jr}^{\perp}, V_r \right\rangle,$$



where $e_{jr}^{\perp}$ is the edge of t_r opposite to vertex j rotated by $\pi/2$, such that it points outside the triangle (see the inset figure for the notations).

Proof The gradient of a basis hat function is given by (see e.g. [?]): $\nabla\gamma_j = e_{jr}^{\perp}/(2\mathcal{A}_r)$, where $\mathcal{A}_r$ is the area of the triangle t_r. This value is constant in t_r, as is V_r, and therefore we have:

$$d_{ij}^r = \int_{t_r} \gamma_i \langle \nabla\gamma_j, V_r \rangle d\mu = \frac{1}{2\mathcal{A}_r} \left\langle e_{jr}^{\perp}, V_r \right\rangle \int_{t_r} \gamma_i d\mu.$$

The integral of a basis hat function on the whole triangle is exactly the volume of a pyramid with basis t_r and height 1. Hence, $\int_{t_r} \gamma_i d\mu = \mathcal{A}_r/3$. Plugging this in d_{ij}^r we get:

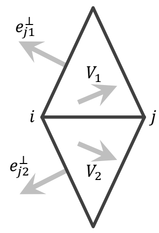
$$d_{ij}^r = \frac{1}{6} \left\langle e_{jr}^{\perp}, V_r \right\rangle.$$

Note, that this expression holds also when $j = i$. □

Now, computing the values of S_{ij} and S_{ii} is simply a matter of identifying on which set of triangles d_{ij}^r is not zero.

For S_{ij}, these are only the two triangles t_1, t_2 neighboring the edge (i,j). Hence we have:

$$S_{ij} = \frac{1}{6} \left(\left\langle e_{j1}^{\perp}, V_1 \right\rangle + \left\langle e_{j2}^{\perp}, V_2 \right\rangle \right),$$



where the notations are given in the inset figure.

For S_{ii}, the relevant triangles are the faces t_r which are near the vertex i (denoted by $N_F(i)$), hence we have:

$$S_{ii} = \frac{1}{6} \sum_{t_r \in N_F(i)} \left\langle e_{ir}^{\perp}, V_r \right\rangle.$$

Finally, we would like to show that $S_{ii} = -\sum_j S_{ij}$. From the definition of S_{ij} we have that:

$$\sum_j S_{ij} = \frac{1}{6} \sum_{j \in N(i)} \left(\left\langle e_{j1}^{\perp}, V_1 \right\rangle + \left\langle e_{j2}^{\perp}, V_2 \right\rangle \right).$$

By re-arranging the sum as a sum on the neighboring faces, we get:

$$\sum_j S_{ij} = \frac{1}{6} \sum_{r=(i,j,k) \in N_F(i)} \left(\left\langle e_{jr}^{\perp}, V_r \right\rangle + \left\langle e_{kr}^{\perp}, V_r \right\rangle \right).$$

It is easy to check that for a triangle $r = (i,j,k)$ we have:

$$e_{jr} + e_{kr} = (p_i - p_k) + (p_j - p_i) = p_j - p_k = -e_{ir},$$

and hence:

$$\sum_j S_{ij} = \frac{1}{6} \sum_{r=(i,j,k)\in N_F(i)} \left(\left\langle -e_{ir}^{\perp}, V_r \right\rangle \right) = -S_{ii}.$$

4.2. Proofs

Lemma 4.1 Let $M=(X,F,N)$ and let V_1,V_2 be two piecewise constant vector fields on M. Then: $\hat{D}^F_{V_1}=\hat{D}^F_{V_2}$ if and only if $V_1=V_2$.

Proof We will show that given a tangent vector field V, and a corresponding operator $\hat{D}^F_V$, we can reconstruct V uniquely from $\hat{D}^F_V$. Since $\hat{D}^F_V$ is defined locally per face, where V is smooth, the uniqueness is in fact implied by the uniqueness property in the smooth case. However, for completeness we will validate this explicitly, by providing a reconstruction method that extracts V given $\hat{D}^F_V$.

Given a face $r=(i,j,k)$ we compute $c_i=(\hat{D}^F_V(\gamma_i))_r$ and similarly for c_j,c_k, where γ_i is the hat basis function of vertex i. Now, we consider the set of constraints we have on V_r. First, by definition we have that $(\hat{D}^F_V(\gamma_i))_r=\langle\nabla\gamma_i,V_r\rangle=c_i$. In addition, V_r should be tangent to the triangle, hence $\langle V_r,N_r\rangle=0$, where N_r is the normal. This yields the following linear system for V_r:

$$\begin{pmatrix} (\nabla\gamma_i)_r^T \\ (\nabla\gamma_j)_r^T \\ (\nabla\gamma_k)_r^T \\ N_r^T \end{pmatrix} V_r = \begin{pmatrix} c_i \\ c_j \\ c_k \\ 0 \end{pmatrix}$$

However, since $s=\gamma_i+\gamma_j+\gamma_k=1$, we have that $\hat{D}^F_V(s)=c_i+c_j+c_k=0$, and similarly $\nabla\gamma_i+\nabla\gamma_j+\nabla\gamma_k=0$. Therefore, one of the equations is redundant. Furthermore, $\nabla\gamma_i$ is in the direction of the edge (j,k) rotated by $\pi/2$, and similarly for $\nabla\gamma_j$ and they are both orthogonal to N_r. Therefore, if the triangle is not degenerate, $\nabla\gamma_i,\nabla\gamma_j,N_r$ are linearly independent, and the system is full rank. Since we know that $\hat{D}^F_V$ was constructed from V, the system has a unique solution given by V_r. □

Lemma 4.2 Let $M_1=(X_1,F,N_1)$ and $M_2=(X_2,F,N_2)$ be two triangle meshes with the same connectivity but different metric (i.e. different embedding). Additionally, let V_1 be a piecewise constant vector field on M_1, then:

$$\hat{D}^F_{V_1}=\hat{D}^F_{V_2}.$$

Here $(V_2)_r=A(V_1)_r$, where A is the linear transformation that takes the triangle r in M_1 to the corresponding triangle in M_2. Note that $\hat{D}_{V_i}$ is computed using the embedding X_i.

Proof By definition we have that

$$(\hat{D}^F_{V_1})_{ri} = \langle(\nabla\gamma_i)_1,(V_1)_r\rangle = \left\langle \frac{R^{90}(p_k^1-p_j^1)}{2\mathcal{A}_1},(V_1)_r \right\rangle,$$

where the face $r=(i,j,k)$, p_i^1 are the coordinates in X_1 of vertex i and R^{90} is counter-clockwise rotation by $\pi/2$ in the plane of the triangle r. On the other hand we have

$$\begin{aligned}(\hat{D}^F_{V_2})_{ri} = \langle(\nabla\gamma_i)_2,(V_2)_r\rangle &= \left\langle \frac{R^{90}(p_k^2-p_j^2)}{2\mathcal{A}_2},(V_2)_r \right\rangle \\ &= \left\langle \frac{R^{90}A(p_k^1-p_j^1)}{2|A|\mathcal{A}_1},A(V_1)_r \right\rangle,\end{aligned}$$

where $|A|$ is the determinant of A. It is easy to check directly, that for any A we have that: $A^T(R^{90})^TA=|A|(R^{90})^T$, which implies $\hat{D}^F_{V_1}=\hat{D}^F_{V_2}$, as required. □

Lemma 4.3 Let $M=(X,F,N)$, V a piecewise constant vector field on M, $f=\sum_i f_i\gamma_i$ a PL function on M, and w_i the Voronoi area weights, then:

$$\sum_{i=1}^{|X|} w_i(\hat{D}_V f)_i = \sum_{i=1}^{|X|} w_i(\mathrm{div}(V))_i f_i.$$

where:

$$(\mathrm{div}(V))_i = \frac{1}{2w_i}\sum_{t_r\in N_F(i)} \left\langle V_r, e_{ir}^{\perp} \right\rangle.$$

Proof From the definition of $\hat{D}_V$, we have that

$$\sum_{i=1}^{|X|} w_i(\hat{D}_V f)_i = \sum_{i=1}^{|X|}(W\hat{D}_V f)_i = \sum_{i=1}^{|X|}(Sf)_i = \sum_{i=1}^{|X|}\sum_{j=1}^{|X|} S_{ij}f_j.$$

Switching the roles of the indices i,j, we get:

$$\sum_{i=1}^{|X|}\sum_{j=1}^{|X|} S_{ji}f_i = \sum_{i=1}^{|X|} g_i f_i, \quad g_i = \sum_{j=1}^{|X|} S_{ji}.$$

The only non-zero entries in the i-th column of S are on the diagonal and entries S_{ji} such that j is a neighbor of i. Thus we have:

$$g_i = S_{ii} + \sum_{j\in N(i)} S_{ji}.$$

Plugging in the definition of S_{ji} and S_{ii} we get:

$$g_i = \frac{1}{6}\sum_{t_r\in N_F(i)} \left\langle e_{ir}^{\perp},V_r \right\rangle + \frac{1}{6}\sum_{j\in N(i)} \left(\left\langle e_{i1}^{\perp},V_1 \right\rangle + \left\langle e_{i2}^{\perp},V_2 \right\rangle \right).$$

Again, we can re-arrange the second sum as a sum on neighboring faces and get:

$$\begin{aligned} g_i &= \frac{1}{6}\sum_{t_r\in N_F(i)} \left\langle e_{ir}^{\perp},V_r \right\rangle + \frac{1}{6}\sum_{t_r\in N_F(i)} \left(\left\langle e_{ir}^{\perp},V_r \right\rangle + \left\langle e_{ir}^{\perp},V_r \right\rangle \right) \\ &= \frac{1}{2}\sum_{t_r\in N_F(i)} \left\langle e_{ir}^{\perp},V_r \right\rangle = w_i(\mathrm{div}(V))_i. \end{aligned}$$

Finally, we get:

$$\sum_{i=1}^{|X|} w_i(\hat{D}_V f)_i = \sum_{i=1}^{|X|} g_i f_i = \sum_{i=1}^{|X|} w_i(\mathrm{div}(V))_i f_i,$$

as required. □

References

[Spi99] SPIVAK M.: *A comprehensive introduction to differential geometry. Vol. I*, third ed. Publish or Perish Inc., 1999.